AE-802

M.A./M.Sc. (Previous)
Term End Examination, 2016-17

MATHEMATICS

Compulsory

Paper - IV

Complex Analysis

Time : Three Hours] [*Maximum Marks* : 100
[*Minimum Pass Marks* : 36

Note : Answer any **five** questions. Answer to each question should begin on a fresh page. All questions carry equal marks.

1. (*a*) If a function $f(z)$ is analytic within and on a closed contour C and a is any point lying in it, then prove that

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$$

(*b*) Obtain the Taylor's series and Laurent series which represent the function $\frac{z^2 - 1}{(z+2)(z+3)}$ in the regions—

(*i*) $2 < |z| < 3$ (*ii*) $|z| > 3$

(*iii*) $|z| < 2$

2. (*a*) State and prove Morera's theorem.

(*b*) Prove that the function $\sin\left\{c\left(z+\frac{1}{z}\right)\right\}$ can be expanded in a series of the type $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ in which the coefficients of both z^n and z^{-n}, are

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta \, d\theta$$

3. (*a*) Find the residues at all singularities in a contour C of the functions $\frac{e^z}{z^3 - z^2}$ and $\frac{z+1}{z^2(z-3)}$.

(*b*) By the method of contour integration, prove that

$$\int_0^{\infty} \frac{dx}{\left(1+x^2\right)^2} = \frac{\pi}{4}$$

4. (*a*) Find all the Mobius transformations which transform the half plane $I(z) \geq 0$ into circles $|w| \leq 1$.

(*b*) If $w = f(z)$ represents a conformal mapping of a domain D in the z-plane into a domain D' of the w-plane, then prove that $f(z)$ is an analytic function of z in D.

5. (*a*) State and prove Montel's theorem.

(*b*) Prove that the space of analytic functions $H(G)$ is complete metric space.

6. (*a*) For $0 < \text{Re } z < 1$, prove that

$$\zeta(z)\sqrt{z} = \int_0^{\infty}\left(\frac{1}{e^t - 1} - \frac{1}{t}\right)t^{z-1}dt$$

(*b*) Define the following terms :

(*i*) Analytic continuation along a chain of domains

(*ii*) Analytic continuation along a curve

(*iii*) Complete analytic function

(*iv*) Natural boundary

(*v*) Function element

7. (*a*) Let f be a function defined on $(0, \infty)$ such that $f(x) > 0$ for all $x > 0$. Suppose that f has the following properties :

(*i*) $\log f(x)$ is a convex function

(*ii*) $f(x+1) = xf(x)$ for all x

(*iii*) $f(1) = 1$

Then prove that $f(x) = \sqrt{x}$ for all x.

(*b*) State and prove Monodromy theorem.

8. (*a*) Suppose $f(z)$ is analytic in a closed ring $r_1 \le |z| \le r_3$. Let $r_1 < r_2 < r_3$ and $M(r_i)$ be the maximum value of $|f(z)|$ on the circles $|z| = r_i$ $(i = 1, 2, 3)$. Then prove that

$$\log M(r_2) \le \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log M(r_3).$$

(*b*) State and prove Jenson's inequality.

9. (*a*) Let f be a non-constant entire function of order λ with $f(0) = 1$ and let $\{a_1, a_2, \ldots\}$ be the zeroes of f counted according to multiplicity and arranged so that $|a_1| \le |a_2| \le \ldots\ldots$

If an integer $p > \lambda - 1$, then prove that

$$\frac{d^p}{dz^p}\left[\frac{f'(z)}{f(z)}\right] = -p! \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{p+1}} \text{ for}$$

$z \ne a_1, a_2 \ldots\ldots$

(*b*) If $|z| \le 1$ and $p \ge 0$, then prove that $|1 - E_p(z)| \le |z|^{p+1}$

where $E_p(z)$ are elementary factors of an analytic function.

10. (*a*) State and prove Little Picard's theorem.

(*b*) Suppose g is analytic on $B(0, R)$, $g(0)=0$, $|g'(0)| = \mu > 0$ and $|g(z)| \le M$ for all z, then prove that

$$g(B(0,R)) \supseteq B\left(0, \frac{R^2\mu^2}{6M}\right)$$

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