AE-805

M.A./M.Sc. (Final)
Term End Examination, 2016-17

MATHEMATICS

Compulsory

Paper - I

Integration Theory and Functional Analysis

Time : Three Hours] [*Maximum Marks* : 100
[*Minimum Pass Marks* : 36

Note : Answer any **five** questions. All questions carry equal marks.

1. (*a*) State and prove Extension theorem.

(*b*) State and prove Hahn decomposition theorem.

2. (*a*) Explain the Baire sets and Baire measure.

(*b*) Let (X, A, μ) be finite measure space, let $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If F is a continuous linear function of $Lp(X)$, then there exists $a, g \in Lq(X)$ such that

$$F(f)=\int_X fg \, d\mu, \quad \text{for all} \quad L \in Lp(X)$$

and $\| F \| = \| g \|_q$. Prove it.

3. (*a*) Show that l_∞, C, C_0 are nls each with the norm $\| x \| = \sup | x_n |$.

Does $\| x \| = \lim_{n \to \infty} | x_n |$ define a norm on C ?

(*b*) A nls X is complete if and only if every absolutely convergent series in X is convergent.

4. (*a*) Show that two equivalent norms on a linear space X induce the same topology on X.

(*b*) State and prove Fubini's theorem.

5. (*a*) State and prove Borel-Lebesgue theorem.

(*b*) Let $\{x_n\}$ be a weakly convergent sequence in a normed space X, i.e. $x_n \xrightarrow{w} x$. Then prove that (*i*) Weak limit of the sequence $\{ x_n \}$ is unique.

(*ii*) Every subsequence of $\{ x_n \}$ convergence weakly to x.

(*iii*) The sequence $\| x_n \|$ is bounded.

6. (*a*) State and prove Hahn-Banach theorem for real linear spaces.

(*b*) Show that the Schwarz inequality in Hilbert space.

7. (*a*) Explain the Gram-Schmidt orthogonalization process.

(*b*) Let $\{e_1, e_2, ...e_n\}$ be a finite orthonormal set in a Hilbert space H and x be any element of H. Then prove that

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \| x \|^2$$

8. (*a*) If $T \in B(x, y)$, then prove that

$T^* \in B(y^*, x^*)$ and $\| T \| = \| T^* \|$.

(*b*) Let X and Y be normed spaces. Then prove that

(*i*) Every compact linear operator $T : X \to Y$ is bounded, hence continuous.

(*ii*) If X is infinite dimensional space, the identity operator $I : X \rightarrow X$ (which is continuous) is not compact.

9. (*a*) State and prove open mapping theorem.

(*b*) Let T be a closed linear map of a Banach space X into a Banach space Y. Then T is continuous. Prove it.

10. (*a*) State and prove closed range theorem.

(*b*) Show that every convergent sequence in a normal linear space a Cauchy sequence, but the converse need not be true.