PF-360

M.A./M.Sc. Mathematics
3rd Semester Examination, Dec., 2022

Compulsory

Paper - I

Integration Theory and Functional Analysis - I

Time : Three Hours] [*Maximum Marks* : 80

Note : Answer from **both** the Sections as directed. The figures in the right-hand margin indicate marks.

Section-A

1. Answer the following questions : 1×10

(*a*) A signed measure μ is said to be totally finite if ....

(*b*) A Borel measurable set is ....

(*c*) If μ and ν are both σ-finite, then they have a ....

(*d*) Every Borel set is σ-bounded if and only if ....

(*e*) A set E in $\hat{S}$ is said to be inner regular with respect to μ if ....

(*f*) The Radon-Nikodym theorem remains true even if ....

(*g*) If μ is signed measure on measurable space (X, A) and $E \subseteq X$ is measurable, then E is positive if ....

(*h*) Hahn decomposition is unique except for ....

(*i*) If ν is signed measure such that $\nu \perp \mu$ and $\nu << \mu$, then the ...... of ν is ....

(*j*) If μ is the Borel measure induced by a regular content λ, then $\mu(C) =$ ...... for every C is ζ.

2. Answer the following questions : 2×5

(*a*) Define section of any subset.

(*b*) Define measurable rectangle.

(*c*) Define Positive set and Negative set.

(*d*) State Lebesgue Decomposition theorem.

(*e*) State Riesz representation theorem.

Section-B

Answer **all** questions : 12×5

3. State and prove Radon-Nikodym theorem.

OR

Let E be a measurable set of finite measure. Then prove that E contains a positive set A with $\mu(A) > 0$.

4. Show that if V_1 and V_2 are any two finite signed measures then so is $\alpha V_1 + \beta V_2$, where α, β are real numbers. Show that

$$|\alpha V| = |\alpha|\,|V| \text{ and } |V_1 + V_2| \le |V_1| + |V_2|$$

where $V \le \mu$ means $V(E) \le \mu(E)$ for all measurable set E.

OR

Show that the collection E^* of all μ^*-measurable sets is σ-algebra containing E. Also if $<A_n>$ is a disjoint sequence in E^* then prove that

$$\mu^*\left[\bigcup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} \mu^*(A_n)$$

5. Using Fubini's theorem verify

$$\int_0^1\left[\int_0^1 \frac{x-y}{(x+y)^3}\,dx\right]dy \ne \int_0^1\left[\int_0^1 \frac{x-y}{(x+y)^3}\,dy\right]dx$$

OR

Prove that necessary and sufficient condition that a function should be an indefinite integral is that it should be absolutely continuous.

6. If $\mathop{\mu}\limits_{\circ}$ is a Baire measure and if for every C in ζ

$$\lambda(C) = \inf\left\{\mathop{\mu}\limits_{\circ}(U_0) : CCU_0 \in U_0\right\}$$

Then λ is a regular content.

OR

Show that the Borel measure μ is not regular.

7. (*a*) Prove that a Borel measurable set is Lebesgue measurable.

(*b*) Prove that every Borel measure is σ-finite.

OR

(*a*) Prove that finite disjoint union of inner regular sets of finite measure is inner regular.

(*b*) If $\int \phi\, d\alpha = \int \phi\, d\beta$ for all $\phi \in L$, then prove that α, β have the same discontinuities.